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BPS branes in discrete torsion orbifolds

Hanno Klemm*

Institut für Theoretische Physik, ETH Hönggerberg
8093 Zürich, Switzerland
and
Department of Mathematics, King's College London,
Strand, London WC2R 2LS, UK

Abstract

We investigate D-branes in a $\mathbb{Z}_3 \times \mathbb{Z}_3$ orbifold with discrete torsion. For this class of orbifolds the only known objects which couple to twisted RR potentials have been non-BPS branes. By using more general gluing conditions we construct here a D-brane which is BPS and couples to RR potentials in the twisted and in the untwisted sectors.

*klemm@phys.ethz.ch

1. Introduction

Orbifolds [1] have been considered extensively in string theory as interesting backgrounds which capture a lot of the geometric information of Calabi-Yau backgrounds while still being sufficiently easy to be treated exactly by CFT methods. Shortly after the introduction of orbifolds, Vafa introduced a slight generalisation of the concept of an orbifold which is called a discrete torsion orbifold [2]. D-branes on orbifolds have been considered in a plethora of contexts, and are by now quite well understood. For some early works see for example [3, 4]. D-branes on orbifolds with discrete torsion have been considered less frequently in the literature and some puzzles still remain. Some previous work on D-branes in discrete torsion orbifolds can be found in [5–18]. D-branes in theories with discrete torsion often give rise to projective representations of the orbifold group on the Chan-Paton factors. However, this feature does not describe discrete torsion branes uniquely, as projective representations can also appear on orbifolds without discrete torsion [16]. In this note we want to address the following open problem of D-branes in discrete torsion orbifolds. If we consider a $\mathbb{Z}_n \times \mathbb{Z}_n$ orbifold with discrete torsion, we can calculate the RR potentials in the theory. Since D-branes act as sources of RR flux, they have to couple to these RR potentials. In the case of a $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold, BPS D-branes have been constructed which couple to all RR potentials. The branes which couple to the twisted RR potentials were given by fractional BPS-branes with one Dirichlet and one Neumann boundary condition along each complex direction in the orbifold [15].

However, for the case n odd the only known D-branes which couple to the twisted RR potentials are non-BPS branes [16]. This is somewhat unusual, since one should expect the branes which carry the minimal charges to be BPS. On a technical level the problem seemed to be that the BPS D-brane which carried the twisted charges in the $\mathbb{Z}_2 \times \mathbb{Z}_2$ case was not compatible with the orbifold action for odd n.

The aim of this note is to construct a BPS brane which couples to twisted RR charges in a $\mathbb{Z}_3 \times \mathbb{Z}_3$ orbifold with discrete torsion by using the boundary state formalism (for reviews see [19–22]). In order to construct D-branes which are BPS in this background, we have to consider more general boundary conditions than the ones previously considered. Namely, we impose permuted gluing conditions that couple left moving fields in direction i, and right moving fields in direction j. Consistency of these gluing conditions under the orbifold group leaves us basically one additional combination of fields which can be glued together. We will then show that this combination leads to a consistent BPS brane.

It might seem natural to interpret the boundary states for this kind of gluing conditions as permutation branes [23, 24] for the U(1) theory. However, in contradistinction to permutation boundary states in $N = 2$ minimal models, which were recently considered in [25,26], the "permuted" gluing conditions in the U(1) case actually turn out to describe D3-branes at angles. This is in a sense similar to the result of [27] where it was shown that all $N = 2$ superconformal boundary states at $c = 3$ can be described in terms of usual Neumann and Dirichlet branes. We will come back to that point in section 6.

This paper is organised as follows. In section 2 we will briefly review the effect of discrete torsion on the construction of orbifolds. In section 3 we will then remind the

reader of the usual construction of D-branes in orbifold theories with discrete torsion, and in the following sections we will construct the well known example of the D0-brane in the $\mathbb{Z}_3 \times \mathbb{Z}_3$ orbifold with and without discrete torsion. In section 6 we will construct a BPS-brane in the $\mathbb{Z}_3 \times \mathbb{Z}_3$ orbifold with discrete torsion and show that it couples to the twisted RR potentials. Section 7 contains our conclusions. We have summarised our conventions on RR zero modes in an appendix.

2. Discrete torsion

Shortly after the introduction of orbifolds, Vafa [2] pointed out the possibility of discrete torsion orbifolds. Let us briefly recall their definition.

Consider the orbifold of a manifold $\mathcal{M}$ by a discrete Abelian group Γ. As is well known [1], the orbifold theory consists of the Γ invariant states of the original theory, and additional twisted sectors that describe strings which are closed only up to a group transformation by an element of Γ. These are strings which are closed only in $\mathcal{M}/\Gamma$ but not in $\mathcal{M}$. In an Abelian orbifold, we have one twisted sector $\mathcal{H}_h$ for each element $h \in \Gamma$. Each twisted sector has again to be projected onto the Γ invariant states. The projector onto group invariant states is given by

$$P_\Gamma = \frac{1}{|\Gamma|} \sum_{g \in \Gamma} g. \tag{2.1}$$

Therefore, the partition function of the theory is given by

$$Z(q, \bar{q}) = \frac{1}{|\Gamma|} \sum_{g,h \in \Gamma} Z(q, \bar{q}; g, h), \tag{2.2}$$

where

$$Z(q, \bar{q}; g, h) = \mathrm{Tr}_{\mathcal{H}_h}(g q^{L_0} \bar{q}^{\tilde{L}_0}). \tag{2.3}$$

The discrete torsion theory is characterised by the fact that the partition function is modified to

$$Z(q, \bar{q}) = \frac{1}{|\Gamma|} \sum_{g,h \in \Gamma} \epsilon(g, h) Z(q, \bar{q}; g, h), \tag{2.4}$$

where $\epsilon(g, h)$ are phases which have to obey the additional consistency requirements [2]

$$\epsilon(g_1 g_2, g_3) = \epsilon(g_2, g_3)\epsilon(g_1, g_3)\,, \tag{2.5}$$

$$\epsilon(g, h) = \epsilon(h, g)^{-1}\,, \tag{2.6}$$

$$\epsilon(g, g) = 1\,. \tag{2.7}$$

The distinct discrete torsion phases are in one-to-one correspondence with the second group cohomology $\mathrm{H}^2(\Gamma,\mathrm{U}(1))$.

We can interpret the appropriate projector for the theory with discrete torsion to be given by

$$P_\Gamma|_{\mathcal{H}_h} = \frac{1}{|\Gamma|} \sum_{g \in \Gamma} \epsilon(g, h) g|_{\mathcal{H}_h}\,. \tag{2.8}$$

Alternatively, we can interpret the theory with discrete torsion as the one where $g \in \Gamma$ acts on the sector $\mathcal{H}_h$ as

$$\hat{g}|_{\mathcal{H}_h} = \epsilon(g,h) g|_{\mathcal{H}_h}. \tag{2.9}$$

Due to the representation property of the discrete torsion phases, this gives a well-defined action of Γ on $\mathcal{H}_h$. From the conformal field theory point of view there is thus no fundamental difference between the theory with discrete torsion and the theory without. The only difference lies in the definition of the action of the orbifold group on the twisted sectors which is *a priori* not well defined. While the action of the orbifold group in the untwisted sector is fixed by the geometrical action of the group, this is not true for the twisted sectors.

3. D-branes on orbifolds with discrete torsion

In this section we want to spell out some properties of D-branes on orbifolds with discrete torsion. There exists quite an extensive literature on these matters, see for example [3, 5, 6, 10, 15, 16, 19, 21, 28–31]. Let us assume that spacetime is a product of Minkowski space and an orbifold. Let us first consider a bulk D-brane at a generic point of the orbifold. If we start with a D-brane in the covering space, we have to add images of this brane, in order to obtain an orbifold invariant configuration. As we are looking at a generic point of the orbifold, we will need $|\Gamma|$ copies of the original brane. The Γ-invariant open strings that end on any two of these branes describe the excitations of the D-branes. The action of Γ on an open string state $|\psi, ij\rangle$ can be decomposed into an action on the oscillators, and an action on the Chan-Paton factors

$$g|\psi, ij\rangle = \gamma(g)_{ii'} |U(g)\psi, i'j'\rangle \gamma(g)^{-1}_{j'j}. \tag{3.1}$$

In the case of a bulk brane the action on the Chan-Paton factors is given by the regular representation of Γ which has indeed dimension $|\Gamma|$.

If we look at a D-brane near a fixed point of the orbifold, the dimension of the representation γ may be smaller. This is a consequence of the fact that fewer preimages of the brane are needed at a fixed point. Branes for which the dimension of γ is strictly smaller than $|\Gamma|$ are called *fractional* D-branes. Fractional branes are confined to reside on the fixed points because they can not move off into the bulk. This is simply a consequence of the fact that the dimension of γ for those branes is smaller than for ordinary bulk branes. To obtain a bulk brane from fractional branes, one has to combine several fractional branes.

By adding D-branes to a given string theory we introduce open strings into a consistent closed string theory. In order to analyse the consistency conditions for the open string sector, it is often useful to compute the cylinder diagram between two (possibly identical) D-branes. This cylinder amplitude has two rather different interpretations, depending on which world-sheet coordinate is given the rôle of the time coordinate. In the open-string picture this diagram is a one-loop vacuum diagram. In the closed string picture it describes the tree-level exchange of closed strings between two D-branes. The condition

that the closed and the open string descriptions should match imposes strong constraints on the possible open string sectors which can be added consistently to a given closed string theory.

In the closed string picture each D-brane can be described as a boundary state $\|D\rangle\rangle$ which is a coherent state in the closed string Hilbert space. The distinction between bulk branes and fractional branes can as well be seen in the boundary state approach. The boundary state of a bulk brane has only components in the untwisted string Hilbert spaces, whereas boundary states which correspond to fractional branes have components in at least one twisted sector, as well. Components of a D-brane in a given sector describe the coupling of the brane to closed strings in that sector. The contribution of the h-twisted sector to the cylinder diagram corresponds in the open string description to an insertion of h in the one-loop diagram. The sum over twisted sectors then reproduces the projection operator (2.1). This implies that boundary states with a non-trivial contribution in the h-twisted sector lead to open strings for which the annulus diagram contains non-trivial contributions in the sector with h insertion.

4. The $\mathbb{Z}_3 \times \mathbb{Z}_3$ orbifold with discrete torsion

We will consider the example of a $\mathbb{Z}_3 \times \mathbb{Z}_3$ orbifold with discrete torsion of a Type II theory on $\mathbb{C}^3$. It is useful to introduce complexified coordinates along the orbifold directions, as the orbifold group then has a natural diagonal action on the coordinate fields. The generators of the orbifold group g_1 and g_2 act as

$$g_1 : (z^1, z^2, z^3) \mapsto (z^1, e^{-2\pi i/3} z^2, e^{2\pi i/3} z^3)\,, \tag{4.1a}$$

$$g_2 : (z^1, z^2, z^3) \mapsto (e^{2\pi i/3} z^1, z^2, e^{-2\pi i/3} z^3)\,. \tag{4.1b}$$

The different possible choices of discrete torsion phases are in one-to-one correspondence with the second cohomology class $\mathrm{H}^2(\Gamma,\mathrm{U}(1))$. It is known that for $\Gamma = \mathbb{Z}_m \times \mathbb{Z}_m$, $\mathrm{H}^2(\Gamma,\mathrm{U}(1))$ is isomorphic to $\mathbb{Z}_m$. Thus in our example the group of different discrete torsion phases is isomorphic to $\mathbb{Z}_3$. We therefore have the possibility to set

$$\omega = \epsilon(g_1, g_2) = \begin{cases} 1\,, \\ e^{2\pi i/3}\,, \\ e^{-2\pi i/3}\,. \end{cases} \tag{4.2}$$

The first choice corresponds to the case which is usually called the theory without discrete torsion. It should be clear that for $\Gamma = \mathbb{Z}_3 \times \mathbb{Z}_3$ fixing $\epsilon(g_1, g_2)$ determines all discrete torsion phases. This is due to the consistency conditions mentioned in section 2.

In order to summarise the RR ground state spectrum, let us state the Hodge diamond for the $\mathbb{Z}_3 \times \mathbb{Z}_3$ orbifold. The untwisted sector contributes

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & 3 & & 0 & \\ 1 & & 0 & & 0 & & 1 \\ & 0 & & 3 & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array} \tag{4.3}$$

to the Hodge diamond [6]. In the theory with $\omega = 1$, the contribution from the twisted sectors is given by

$$\begin{array}{ccccccc} & & & 0 & & & \\ & & 0 & & 0 & & \\ & 0 & & 7 & & 0 & \\ 0 & & 0 & & 0 & & 0 \\ & 0 & & 7 & & 0 & \\ & & 0 & & 0 & & \\ & & & 0 & & & \end{array} \tag{4.4}$$

For $\omega \neq 1$ we obtain on the other hand

$$\begin{array}{ccccccc} & & & 0 & & & \\ & & 0 & & 0 & & \\ & 0 & & 0 & & 0 & \\ 0 & & 3 & & 3 & & 0 \\ & 0 & & 0 & & 0 & \\ & & 0 & & 0 & & \\ & & & 0 & & & \end{array} \tag{4.5}$$

In the theory with $\omega \neq 1$, the BPS branes constructed in the literature [6, 19] do not couple to these twisted RR potentials. The only D-brane hitherto constructed which couples to the above twisted RR potentials for $\omega \neq 1$ is non-BPS [16], and does not couple to the untwisted RR potentials. In the sequel we will construct a BPS D-brane which couples to all of these potentials.

In order to construct this brane, we will first fix our conventions by constructing a fractional $D(r, 0)$-brane in the theory with $\omega = 1$. We denote by a $D(r, s)$-brane a Dp-brane with r Neumann boundary conditions in the directions unaffected by the orbifold, and s Neumann directions along the orbifold, where $p = r + s$. This brane has been constructed before in [19] and we will repeat the analysis here, mainly to fix some conventions and phases. The fractional $D(r, 0)$-brane couples to all twisted sectors.

5. The $D(r,0)$-brane

5.1. The untwisted sector

We will now construct the boundary state of the $D(r,0)$-brane with $\omega = \epsilon(g_1, g_2) = 1$. This section is basically a review of the work in [19], and we need it to introduce some notation and conventions.

The $D(r,0)$-brane has Dirichlet gluing conditions in all spatial directions along the orbifold and r Neumann gluing conditions in the other directions. With this notation it is possible to treat type IIA and type IIB theories on an equal footing. The number of Neumann conditions along the directions unaffected by the orbifold can then be chosen in accordance with the GSO projection [29].

For concreteness we will consider the directions $2,\ldots,7$ to be the directions along which the orbifold acts. The gluing conditions are given by

$$(a^i_m - \tilde{a}^i_{-m})|D(r,0)\rangle\rangle = 0\,, \qquad i = 2,\ldots,7\,. \tag{5.1}$$

For the sake of simplicity, we will work in light cone gauge by taking $x^8 \pm x^9$ as light-cone coordinates, after a double Wick rotation on x^0 and x^8 [32]. As has been mentioned in the previous chapter, it is convenient to group the directions which are affected by the orbifold action into complex pairs.

The bosonic fields of the closed string are given by

$$X^\mu = x^\mu + 2\pi p^\mu t + \frac{i}{\sqrt{2}}\sum_{n\neq 0}\frac{1}{n}(a^\mu_n e^{-2\pi i n(t-\sigma)} + \tilde{a}^\mu_n e^{-2\pi i n(t+\sigma)})\,,$$

and the corresponding fermionic fields are given by

$$\lambda^\mu = \sqrt{2\pi}\sum_r \lambda^\mu_r e^{-2\pi i r(t-\sigma)}\,,$$
$$\tilde{\lambda}^\mu = \sqrt{2\pi}\sum_r \tilde{\lambda}^\mu_r e^{-2\pi i r(t+\sigma)}\,.$$

If we group these into complex pairs, we obtain

$$Z^i := \frac{1}{\sqrt{2}}(X^{2i} + iX^{2i+1})\,, \qquad Z^{\bar{\imath}} := \frac{1}{\sqrt{2}}(X^{2i} - iX^{2i+1})\,, \qquad i = 1,2,3$$
$$\psi^i := \frac{1}{\sqrt{2}}(\lambda^{2i} + i\lambda^{2i+1})\,, \qquad \psi^{\bar{\imath}} := \frac{1}{\sqrt{2}}(\lambda^{2i} - i\lambda^{2i+1})\,.$$

On the fields Z^i (ψ^i) the orbifold group acts as given in equation (4.1), and on the $Z^{\bar{\imath}}$ ($\psi^{\bar{\imath}}$) it acts with the opposite phases.

On the level of the oscillators we obtain

$$\alpha^i_n = \frac{1}{\sqrt{2}}(a^{2i}_n + ia^{2i+1}_n)\,, \qquad i = 1,2,3$$
$$\alpha^{\bar{\imath}}_n = \frac{1}{\sqrt{2}}(a^{2i}_n - ia^{2i+1}_n)\,,$$

where we adopt the same conventions for the right-movers, and analogously for the fermionic modes we write

$$\psi_r^i = \frac{1}{\sqrt{2}}(\lambda_r^{2i} + i\lambda_r^{2i+1})\,,$$
$$\psi_r^{\bar{\imath}} = \frac{1}{\sqrt{2}}(\lambda_r^{2i} - i\lambda_r^{2i+1})\,.$$

The (anti-)commutation relations for the complexified oscillators are given by

$$[\alpha_n^i, \alpha_m^{\bar{\jmath}}] = [\tilde{\alpha}_n^i, \tilde{\alpha}_m^{\bar{\jmath}}] = n\delta_{n+m}\delta^{ij}\,,$$
$$\{\psi_r^i, \psi_s^{\bar{\jmath}}\} = \{\tilde{\psi}_r^i, \tilde{\psi}_s^{\bar{\jmath}}\} = \delta_{r+s}\delta^{ij}\,.$$

In terms of complexified coordinates along the orbifold, the gluing conditions are given by

$$(\alpha_n^i - \tilde{\alpha}_{-n}^i)|D(r,0)\rangle\!\rangle = 0\,, \tag{5.2a}$$
$$(\alpha_n^{\bar{\imath}} - \tilde{\alpha}_{-n}^{\bar{\imath}})|D(r,0)\rangle\!\rangle = 0\,, \tag{5.2b}$$
$$(\psi_r^i - i\eta\tilde{\psi}_{-r}^i)|D(r,0),\eta\rangle\!\rangle = 0\,, \tag{5.2c}$$
$$(\psi_r^{\bar{\imath}} - i\eta\tilde{\psi}_{-r}^{\bar{\imath}})|D(r,0),\eta\rangle\!\rangle = 0\,. \tag{5.2d}$$

Here $\eta = \pm 1$ and denotes the possible spin structures on the world-sheet [22].

The NS-NS and RR Ishibashi states which solve the above gluing conditions are the coherent states

$$\begin{aligned}|D(r,0),\eta\rangle\!\rangle_{\substack{NSNS\\RR}} = |Dr\rangle\!\rangle \exp\Bigg[\sum_{n=1}^{\infty}\frac{1}{n}\left(\sum_{i=1}^{3}\alpha_{-n}^i\tilde{\alpha}_{-n}^{\bar{\imath}} + \alpha_{-n}^{\bar{\imath}}\tilde{\alpha}_{-n}^i\right) + \\ + i\eta\sum_{r>0}\sum_{i=1}^{3}\psi_{-r}^i\tilde{\psi}_{-r}^{\bar{\imath}} + \psi_{-r}^{\bar{\imath}}\tilde{\psi}_{-r}^i\Bigg]|0,\eta\rangle_{\substack{NSNS\\RR}}\,.\end{aligned} \tag{5.3}$$

We have denoted by $|Dr\rangle\!\rangle$ the components of the coherent state in the directions which were not orbifolded. In the NS-NS sector $|0,\eta\rangle_{NSNS}$ is the unique NS-NS vacuum, whereas in the RR sector we have to work a little harder. The RR ground states have to solve (5.2) for the fermionic zero modes. Let us define the ψ_0^is to be raising operators, and the $\psi_0^{\bar{\imath}}$s to be lowering operators. We denote the state which is annihilated by $\psi_0^{\bar{\imath}}$ with $|-\rangle$ and the state $\psi_0^i|-\rangle = |+\rangle$. We will adopt the same definition for the right-movers, as well. The ordering conventions for the fermionic zero modes are given in the appendix. Gluing conditions of the form (5.2) are solved by the state $|+\rangle\otimes|-\rangle - i\eta|-\rangle\otimes|+\rangle$. The next step is to build the tensor product of the ground states of the various directions. The ground state which satisfies all gluing conditions is thus given by

$$\prod_{i=1}^{3}(\psi_0^i - i\eta\tilde{\psi}_0^i)|0\rangle_{RR}\,, \qquad \text{with} \qquad |0\rangle_{RR} = |---\rangle\otimes|---\rangle\,. \tag{5.4}$$

Here we only considered the gluing conditions and the RR ground state in the orbifold directions. For the other directions, we can simply tensor the appropriate RR ground state with the above state, once we have decided on the gluing conditions.

We still have to impose the GSO projection. In the NS-NS sector the vacuum is odd under $(-1)^F$ and $(-1)^{\tilde{F}}$ and therefore we obtain

$$(-1)^F |D(r,0),\eta\rangle\rangle_{NSNS} = (-1)^{\tilde{F}} |D(r,0),\eta\rangle\rangle_{NSNS} = -|D(r,0),-\eta\rangle\rangle_{NSNS} \,.$$

In the RR sector the GSO operators act as the chirality operators on the ground state, and for a general Dp-brane we define $|Dp,-\eta\rangle\rangle_{RR}$ by

$$(-1)^F |Dp,\eta\rangle\rangle_{RR} := |Dp,-\eta\rangle\rangle_{RR} \,. \tag{5.5}$$

This fixes the constant of proportionality. Then the action of $(-1)^{\tilde{F}}$ on this state is given by

$$(-1)^{\tilde{F}} |Dp,\eta\rangle\rangle_{RR} = (-1)^{p+1} |Dp,-\eta\rangle\rangle_{RR} \,, \tag{5.6}$$

where $p = r + s$. Therefore the GSO invariant combinations are given by

$$\|D(r,0)\rangle\rangle_{NSNS} = \frac{1}{\sqrt{2}}\big(|D(r,0),+\rangle\rangle_{NSNS} - |D(r,0),-\rangle\rangle_{NSNS}\big)$$

for the NS-NS sector, and for the RR sector of the $D(r,0)$-brane we obtain

$$\|D(r,0)\rangle\rangle_{RR} = \frac{1}{\sqrt{2}}\big(|D(r,0),+\rangle\rangle_{RR} + |D(r,0),-\rangle\rangle_{RR}\big) \,. \tag{5.7}$$

(5.7) is invariant under the GSO projection of the type IIA theory if r is even, and under the GSO projection of the type IIB theory if r is odd.

In the next step let us see how the boundary state (5.3) behaves under the action of the orbifold group. The oscillators are mapped according to (4.1) to

$$g_k \alpha_n^i g_k^{-1} = e^{2\pi i \nu_k^i} \alpha_n^i \,, \qquad g_k \alpha_n^{\bar{\imath}} g_k^{-1} = e^{-2\pi i \nu_k^i} \alpha_n^{\bar{\imath}}, \tag{5.8a}$$

$$g_k \psi_n^i g_k^{-1} = e^{2\pi i \nu_k^i} \psi_n^i \,, \qquad g_k \psi_n^{\bar{\imath}} g_k^{-1} = e^{-2\pi i \nu_k^i} \psi_n^{\bar{\imath}}, \tag{5.8b}$$

where $\nu_1 = (0,-1/3,1/3)$ and $\nu_2 = (1/3,0,-1/3)$, and similarly for the right-movers. Therefore the terms in the exponential of (5.3) are invariant under the group action. Since the orbifold action on the untwisted NS-NS vacuum is trivial, we only have to determine the action of the orbifold group on the RR ground state. The action on the RR ground state is given by the action on the oscillators and the action on $|0\rangle_{RR}$. It is easy to see that the combinations of oscillators which appear if we expand (5.4) are invariant under Γ. The orbifold action maps annihilation operators again to annihilation operators, therefore a state which has been annihilated by all annihilators will still be destroyed by all annihilation operators after the action of the orbifold group. Therefore $|0\rangle_{RR}$ must carry some representation of the orbifold group,

$$g_k^n |0\rangle_{RR} = \rho_k^n |0\rangle_{RR} \,.$$

The only possible values for ρ are third roots of unity. We know on the other hand that the RR ground state of the $D(r,0)$-brane is not projected out by the orbifold projection [19] and therefore we can fix $\rho_k = 1$.

5.2. The twisted sector

Let us now turn to the twisted sectors. In the twisted sectors the oscillator moding changes since the string only closes up to a group transformation. For the orbifold action in question, the mode expansions for the sector twisted by g_k^m are given by

$$
\begin{aligned}
Z^i &= \frac{i}{\sqrt{2}}\left(\sum_{r\in\mathbb{Z}+m\nu_k^i}\frac{1}{r}\alpha_r^i e^{-2\pi i r(t-\sigma)} + \sum_{r\in\mathbb{Z}-m\nu_k^i}\frac{1}{r}\tilde{\alpha}_r^i e^{-2\pi i r(t+\sigma)}\right),\\
Z^{\bar{\imath}} &= \frac{i}{\sqrt{2}}\left(\sum_{r\in\mathbb{Z}-m\nu_k^i}\frac{1}{r}\alpha_r^{\bar{\imath}} e^{-2\pi i r(t-\sigma)} + \sum_{r\in\mathbb{Z}+m\nu_k^i}\frac{1}{r}\tilde{\alpha}_r^{\bar{\imath}} e^{-2\pi i r(t+\sigma)}\right),\\
\psi^i &= \sqrt{2\pi}\sum_{r\in\mathbb{Z}+m\nu_k^i+s}\psi_r^i e^{-2\pi i r(t-\sigma)},\\
\psi^{\bar{\imath}} &= \sqrt{2\pi}\sum_{r\in\mathbb{Z}-m\nu_k^i+s}\psi_r^{\bar{\imath}} e^{-2\pi i r(t-\sigma)},\\
\tilde{\psi}^i &= \sqrt{2\pi}\sum_{r\in\mathbb{Z}-m\nu_k^i+s}\tilde{\psi}_r^i e^{-2\pi i r(t+\sigma)},\\
\tilde{\psi}^{\bar{\imath}} &= \sqrt{2\pi}\sum_{r\in\mathbb{Z}+m\nu_k^i+s}\tilde{\psi}_r^{\bar{\imath}} e^{-2\pi i r(t+\sigma)},
\end{aligned}
$$

where the ν_k^i are as defined below equation (5.8), and $s=\frac{1}{2}$ ($s=0$) in the NS (R) sector. Note that the mode shifts of the right-movers are opposite to the mode shifts of the left-movers. This is due to the fact that the mode shifts are determined by the behaviour under the transformation $\sigma\to\sigma+1$ which differs by a minus sign between left-movers and right-movers. The (anti-)commutation relations for the twisted oscillators are given by

$$[\alpha_{n+\nu}^i,\alpha_{m-\nu}^{\bar{\jmath}}]=[\tilde{\alpha}_{n+\nu}^i,\tilde{\alpha}_{m-\nu}^{\bar{\jmath}}]=(n+\nu)\delta_{n+m}\,\delta^{ij}, \tag{5.9}$$
$$\{\psi_{r+\nu}^i,\psi_{s-\nu}^{\bar{\jmath}}\}=\{\tilde{\psi}_{r+\nu}^i,\tilde{\psi}_{s-\nu}^{\bar{\jmath}}\}=\delta_{r+s}\,\delta^{ij}, \tag{5.10}$$

and the other relations vanish. The gluing conditions in the g_k^m-twisted sector are those given by (5.2) with the shifted modes in the orbifold directions

$$(\alpha_{n+m\nu_k^i}^i-\tilde{\alpha}_{-n-m\nu_k^i}^i)|D(r,0)\rangle\rangle=0\,, \tag{5.11a}$$
$$(\alpha_{n-m\nu_k^i}^{\bar{\imath}}-\tilde{\alpha}_{-n+m\nu_k^i}^{\bar{\imath}})|D(r,0)\rangle\rangle=0\,, \tag{5.11b}$$
$$(\psi_{r+m\nu_k^i}^i-i\eta\tilde{\psi}_{-r-m\nu_k^i}^i)|D(r,0),\eta\rangle\rangle=0\,, \tag{5.11c}$$
$$(\psi_{r-m\nu_k^i}^{\bar{\imath}}-i\eta\tilde{\psi}_{-r+m\nu_k^i}^{\bar{\imath}})|D(r,0),\eta\rangle\rangle=0\,. \tag{5.11d}$$

Therefore the Ishibashi states in this sector are given by the coherent states

$$\begin{aligned}
&|D(r,0),\eta\rangle\rangle_{\substack{NSNS\\RR},T_{g_k^m}} = \\
&|Dr\rangle\rangle \exp\left[\sum_n\sum_{i=1}^{3}\left(\frac{1}{n-m\nu_k^i}\alpha^i_{-n+m\nu_k^i}\tilde{\alpha}^{\bar{i}}_{-n+m\nu_k^i}+\frac{1}{n+m\nu_k^i}\alpha^{\bar{i}}_{-n-m\nu_k^i}\tilde{\alpha}^{i}_{-n-m\nu_k^i}\right)+\right. \\
&\left.\quad + i\eta\sum_r\sum_{i=1}^{3}\left(\psi^i_{-r+m\nu_k^i}\tilde{\psi}^{\bar{i}}_{-r+m\nu_k^i}+\psi^{\bar{i}}_{-r-m\nu_k^i}\tilde{\psi}^{i}_{-r-m\nu_k^i}\right)\right]|0,\eta\rangle_{\substack{NSNS\\RR},T_{g_k^m}}\,, \qquad (5.12)
\end{aligned}$$

where the summation over n and r should start appropriately, such that all creation operators appear in the exponential. For simplicity, and in an attempt to clutter the notation not any further, we have only written down a sector twisted by g_k^m. There are of course also sectors in the complete theory which are twisted by $g_1^m g_2^l$ and we thus obtain a total of eight twisted sectors. In the additional twisted sectors the reasoning is identical, and we have not explicitly written them down here.

Now we have to turn our attention to the Ramond ground state. In the g_1-twisted sector we have zero modes in the complex 1-plane, and in the directions unaffected by the orbifold. We have again to determine the action of the orbifold group on the relevant zero modes. The gluing conditions for the zero modes are as before, and therefore the state which solves the gluing conditions is given by

$$|0,\eta\rangle_{RR,T_{g_1^m}} = (\psi_0^1 - i\eta\tilde{\psi}_0^1)|-\rangle\otimes|-\rangle\,. \qquad (5.13)$$

As before, we have to determine the action of g_2 on this ground state. (The action of g_1 is of course trivial in the 1-direction.) We can do so as follows: the action of g_2 on $|0,\eta\rangle_{RR,T_{g_1^m}}$ is given by

$$g_2^n|0,\eta\rangle_{RR,T_{g_1^m}} = \omega^{mn}\zeta^n\rho_1^n|0,\eta\rangle_{RR,T_{g_1^m}}\,,$$

where $\zeta = e^{2\pi i/3}$ and we have parametrised the action of g_2 on $|-\rangle\otimes|-\rangle$ by ρ_1. As before, we know from previous work that this state is not projected out in the theory with $\omega = 1$. Therefore, we have to choose $\rho_1 = \bar{\zeta}$. The construction for the g_2 and g_1g_2 twisted sectors is identical. The $g_1g_2^2$-twisted sector does not possess any zero modes along the orbifold directions. As before in the untwisted sector, we have to combine Ishibashi states with different values of η in order to obtain a GSO invariant boundary state. The complete boundary state of the fractional $D(r,0)$-brane is thus a sum of the boundary states of the various twisted sectors,

$$\|D(r,0)\rangle\rangle = \frac{1}{3}\mathcal{N}\sum_{m,n=0}^{2}\|D(r,0)\rangle\rangle_{T_{g_1^m g_2^n}}\,. \qquad (5.14)$$

In order to fix the normalisation, the next step would be to compute the overlap for this brane and compare it with the open string result. This has been done very explicitly in [19] and we will not repeat this calculation here.

6. $\mathbb{C}^3/\mathbb{Z}_3 \times \mathbb{Z}_3$ with discrete torsion

Let us now consider the same orbifold with $\omega \neq 1$. Since discrete torsion only alters the group action in the twisted sectors, the analysis of the untwisted sector remains unchanged. However, as can easily be seen from the discussion of the twisted sector ground states, the RR ground states of the $D(r,0)$-brane will be projected out if we introduce additional discrete torsion phases. Therefore the $D(r,0)$-brane does not couple to the twisted RR potentials. The Hodge diamond (4.5) shows that there exist twisted RR potentials in the theory with $\omega \neq 1$. Since the $D(r,0)$-brane does not couple to them, we have to construct some other brane which is charged under these twisted RR potentials. In [16] it was shown that it is possible to construct non-BPS branes with one Neumann direction along the orbifold, say the 1-direction, and with components only in the g_1-twisted sectors that couple to the twisted RR potentials. This brane however does not couple to any untwisted RR potentials.

6.1. The untwisted sector

We want to consider more general gluing conditions than the ones which have been considered in previous works. By doing so, we will be able to construct a BPS-brane coupling to twisted and untwisted RR potentials. Namely, we want to consider gluing conditions where we impose one Neumann and one Dirichlet condition along the complex 1-direction as has been done before, but we want to permute the gluing in the 2 and 3-directions. In the end, we want to keep Ishibashi states in the g_1-twisted sector, therefore we have to make sure that the gluing conditions imposed can be satisfied by g_1-twisted oscillators. Considering the action of g_1 along the 2 and 3 direction thus leaves us only with one new choice of gluing conditions, namely we can impose

$$(\alpha_r^1 + \tilde{\alpha}_{-r}^{\bar{1}})\|B\rangle\rangle = 0\,, \tag{6.1a}$$

$$(\alpha_r^{\bar{1}} + \tilde{\alpha}_{-r}^{1})\|B\rangle\rangle = 0\,, \tag{6.1b}$$

$$(\alpha_r^2 + \tilde{\alpha}_{-r}^{\bar{3}})\|B\rangle\rangle = 0\,, \tag{6.1c}$$

$$(\alpha_r^{\bar{2}} + \tilde{\alpha}_{-r}^{3})\|B\rangle\rangle = 0\,, \tag{6.1d}$$

$$(\alpha_r^3 + \tilde{\alpha}_{-r}^{\bar{2}})\|B\rangle\rangle = 0\,, \tag{6.1e}$$

$$(\alpha_r^{\bar{3}} + \tilde{\alpha}_{-r}^{2})\|B\rangle\rangle = 0\,, \tag{6.1f}$$

for the bosons and analogously

$$(\psi_r^1 + i\eta\tilde{\psi}_{-r}^{\bar{1}})\|B\rangle\rangle = 0\,, \tag{6.2a}$$

$$(\psi_r^{\bar{1}} + i\eta\tilde{\psi}_{-r}^{1})\|B\rangle\rangle = 0\,, \tag{6.2b}$$

$$(\psi_r^2 + i\eta\tilde{\psi}_{-r}^{\bar{3}})\|B\rangle\rangle = 0\,, \tag{6.2c}$$

$$(\psi_r^{\bar{2}} + i\eta\tilde{\psi}_{-r}^{3})\|B\rangle\rangle = 0\,, \tag{6.2d}$$

$$(\psi_r^3 + i\eta\tilde{\psi}_{-r}^{\bar{2}})\|B\rangle\rangle = 0\,, \tag{6.2e}$$

$$(\psi_r^{\bar{3}} + i\eta\tilde{\psi}_{-r}^{2})\|B\rangle\rangle = 0\,, \tag{6.2f}$$

for the fermions along the orbifold directions. The directions unaffected by the orbifold remain as before. We could replace pairs of plus signs with minus signs in the gluing conditions (6.1) and (6.2). For example we could instead of (6.1c) and (6.1d) impose

$$(\alpha^2_r - \tilde{\alpha}^{\bar{3}}_{-r})\|B\rangle\!\rangle = 0\,, \qquad \text{and} \qquad (\alpha^{\bar{2}}_r - \tilde{\alpha}^{3}_{-r})\|B\rangle\!\rangle = 0\,.$$

However, this choice only determines which of the real directions are glued together with Neumann, and which with Dirichlet boundary conditions. If we try to change only one of the signs, the gluing conditions cease to commute.

If we compare the gluing conditions (6.1) and (6.2) with the action of g_1 on the oscillators, we see that they are compatible with twisting by g_1. The action of g_2 introduces phases on these gluing conditions. Therefore, in order to construct a brane which could possibly be invariant under the orbifold group with the above gluing conditions, the following system of gluing conditions has to be considered.

$$(\alpha^1_r + \zeta^n \tilde{\alpha}^{\bar{1}}_{-r})|B_n\rangle\!\rangle = 0\,, \tag{6.3a}$$
$$(\alpha^{\bar{1}}_r + \bar{\zeta}^n \tilde{\alpha}^{1}_{-r})|B_n\rangle\!\rangle = 0\,, \tag{6.3b}$$
$$(\alpha^2_r + \zeta^n \tilde{\alpha}^{\bar{3}}_{-r})|B_n\rangle\!\rangle = 0\,, \tag{6.3c}$$
$$(\alpha^{\bar{2}}_r + \bar{\zeta}^n \tilde{\alpha}^{3}_{-r})|B_n\rangle\!\rangle = 0\,, \tag{6.3d}$$
$$(\alpha^3_r + \zeta^n \tilde{\alpha}^{\bar{2}}_{-r})|B_n\rangle\!\rangle = 0\,, \tag{6.3e}$$
$$(\alpha^{\bar{3}}_r + \bar{\zeta}^n \tilde{\alpha}^{2}_{-r})|B_n\rangle\!\rangle = 0\,. \tag{6.3f}$$

We impose analogous gluing conditions for the fermions, where we have again set $\zeta = e^{2\pi i/3}$, such that in particular $\zeta^3 = 1$. With these gluing conditions we obtain three Ishibashi states $|B_n\rangle\!\rangle$ such that $g_2|B_n\rangle\!\rangle = |B_{n+1}\rangle\!\rangle$, where we identify $|B_3\rangle\!\rangle$ with $|B_0\rangle\!\rangle$, more generally we take $n \mod 3$. The brane in question is then the orbifold invariant sum of these Ishibashi states.

It is possible to use these gluing conditions to obtain an orbifold invariant, supersymmetric boundary state, as we will see in the following. In the case at hand we find it convenient to use the name permutation brane for the brane we are constructing, but this object does not fall into a new class of D-branes. This is fundamentally different from the case of tensor products of $N = 2$ minimal models for example, where permutation branes are often new objects.

The Ishibashi states which solve the gluing equations (6.1) and (6.3) in the untwisted sector are given by

$$\begin{aligned}
|B_n\rangle\!\rangle_{\substack{NSNS\\RR}} = |Dr\rangle\!\rangle \exp\Bigg[&-\sum_{k=1}^{\infty}\frac{1}{k}\bigg(\bar{\zeta}^n\left(\alpha^1_{-k}\tilde{\alpha}^1_{-k} + \alpha^2_{-k}\tilde{\alpha}^3_{-k} + \alpha^3_{-k}\tilde{\alpha}^2_{-k}\right) \\
&+ \zeta^n\left(\alpha^{\bar{1}}_{-k}\tilde{\alpha}^{\bar{1}}_{-k} + \alpha^{\bar{2}}_{-k}\tilde{\alpha}^{\bar{3}}_{-k} + \alpha^{\bar{3}}_{-k}\tilde{\alpha}^{\bar{2}}_{-k}\right)\bigg) - i\eta\sum_{r>0}\bigg(\bar{\zeta}^n\left(\psi^1_{-r}\tilde{\psi}^1_{-r} + \psi^2_{-r}\tilde{\psi}^3_{-r} + \psi^3_{-r}\tilde{\psi}^2_{-r}\right) \\
&+ \zeta^n\left(\psi^{\bar{1}}_{-r}\tilde{\psi}^{\bar{1}}_{-r} + \psi^{\bar{2}}_{-r}\tilde{\psi}^{\bar{3}}_{-r} + \psi^{\bar{3}}_{-r}\tilde{\psi}^{\bar{2}}_{-r}\right)\bigg)\Bigg]|0,\eta\rangle_{\substack{NSNS\\RR}}\,.
\end{aligned} \tag{6.4}$$

One easily shows that $g_2|B_n\rangle\rangle = |B_{n+1}\rangle\rangle$ and $g_1|B_n\rangle\rangle = |B_n\rangle\rangle$. The orbifold invariant combination is therefore given by

$$\|B\rangle\rangle = \mathcal{N}\sum_{n=0}^{2}|B_n\rangle\rangle\,,$$

where

$$|B_n\rangle\rangle = \tfrac{1}{2}\left(|B_n,+\rangle\rangle_{NSNS} - |B_n,-\rangle\rangle_{NSNS} + \epsilon(|B_n,+\rangle\rangle_{RR} + |B_n,-\rangle\rangle_{RR}\right)\,.$$

In the previous equation ϵ is a sign and distinguishes between branes and anti-branes. In order to simplify the equations we will set $\epsilon = 1$. Let us now check if this brane does couple to RR potentials. To this end, we have to consider the Ramond ground states of the brane system. Analogously to the $D(r,0)$-brane, we can construct the ground states of these branes. In this case they are given by

$$|B_n\rangle\rangle^{(0)}_{RR} = (\psi^1_0\tilde{\psi}^1_0 + i\eta\zeta^n)(\psi^2_0\tilde{\psi}^3_0 + i\eta\zeta^n)(\psi^3_0\tilde{\psi}^2_0 + i\eta\zeta^n)|---\rangle\otimes|---\rangle\,. \tag{6.5}$$

If we expand these states out and take the sum over n, the ground state of $\|B\rangle\rangle$ is given by

$$\|B\rangle\rangle^{(0)}_{RR} = |+++\rangle\otimes|+++\rangle - i\eta|---\rangle\otimes|---\rangle\,, \tag{6.6}$$

where we have, as before, only considered the RR ground states along the orbifold. Therefore, this D-brane does indeed couple to RR potentials in the untwisted sector.

6.2. The twisted sector

Let us now turn to the construction of the twisted sector boundary states. The construction of the g_1-twisted sector is basically identical to the construction of the twisted sector for the $D(r,0)$-brane. The gluing conditions are changed by the mode shifts, and the Ishibashi states in the g_1^m-twisted sectors are then given by

$$\begin{aligned}
&|B_n\rangle\rangle_{\substack{NSNS\\RR},T_{g_1^m}} = \\
&|Dr\rangle\rangle\exp\Bigg[-\sum_k\Bigg(\bar{\zeta}^n\left(\tfrac{1}{k}\alpha^1_{-k}\tilde{\alpha}^1_{-k} + \tfrac{1}{k+m/3}\alpha^2_{-k-m/3}\tilde{\alpha}^3_{-k-m/3} + \tfrac{1}{k-m/3}\alpha^3_{-k+m/3}\tilde{\alpha}^2_{-k+m/3}\right) + \\
&\qquad\zeta^n\left(\tfrac{1}{k}\alpha^{\bar{1}}_{-k}\tilde{\alpha}^{\bar{1}}_{-k} + \tfrac{1}{k-m/3}\alpha^{\bar{2}}_{-k+m/3}\tilde{\alpha}^{\bar{3}}_{-k+m/3} + \tfrac{1}{k+m/3}\alpha^{\bar{3}}_{-k-m/3}\tilde{\alpha}^{\bar{2}}_{-k-m/3}\right)\Bigg) \\
&\qquad - i\eta\sum_r\Bigg(\bar{\zeta}^n\left(\psi^1_{-r}\tilde{\psi}^1_{-r} + \psi^2_{-r-m/3}\tilde{\psi}^3_{-r-m/3} + \psi^3_{-r+m/3}\tilde{\psi}^2_{-r+m/3}\right) \\
&\qquad + \zeta^n\left(\psi^{\bar{1}}_{-r}\tilde{\psi}^{\bar{1}}_{-r} + \psi^{\bar{2}}_{-r+m/3}\tilde{\psi}^{\bar{3}}_{-r+m/3} + \psi^{\bar{3}}_{-r-m/3}\tilde{\psi}^{\bar{2}}_{-r-m/3}\right)\Bigg)\Bigg]|0,\eta\rangle_{\substack{NSNS\\RR}}\,.
\end{aligned} \tag{6.7}$$

Again, the summation should run over all creation operators. The form of these Ishibashi states implies that

$$g_2^l|B_n\rangle\rangle_{NSNS,T_{g_1^m}} = \omega^{lm}|B_{n+l}\rangle\rangle_{NSNS,T_{g_1^m}}\,.$$

We have incorporated discrete torsion by modifying the action of g_2 on the ground state of the g_1^m-twisted sector. It is now defined as

$$g_2^l|0,\eta\rangle_{NSNS,T_{g_1^m}} = \omega^{lm}|0,\eta\rangle_{NSNS,T_{g_1^m}} .$$

Therefore, an orbifold invariant combination of these Ishibashi states is given by

$$\|B\rangle\rangle_{T_{g_1^m}} = \mathcal{N}\sum_{n=0}^{2}\omega^{mn}|B_n\rangle\rangle_{T_{g_1^m}} . \tag{6.8}$$

Let us now turn our attention to the RR ground states in the twisted sector. The gluing conditions are given by

$$\begin{aligned}(\psi_0^1 + i\eta\zeta^n\tilde{\psi}_0^{\bar{1}})|B_n\rangle\rangle^{(0)}_{T_{g_1}} &= 0\,,\\ (\psi_0^{\bar{1}} + i\eta\bar{\zeta}^n\tilde{\psi}_0^{1})|B_n\rangle\rangle^{(0)}_{T_{g_1}} &= 0\,.\end{aligned}$$

We omit the gluing conditions in the directions unaffected by the orbifold, as they are irrelevant for this problem. These gluing conditions can be solved by states

$$|B_n\rangle\rangle^{(0)}_{T_{g_1}} \sim \bar{\zeta}^n|+\rangle\otimes|+\rangle + i\eta|-\rangle\otimes|-\rangle\,,$$

up to normalisations. The action of g_2 on these states is given by the action of g_2 on the oscillators, and the action on the state we have defined as our ground state, namely the state $|-\rangle\otimes|-\rangle$. From the discussion of the $D(r,0)$-brane we know that the action of g_2 therefore is given by

$$g_2^k(\bar{\zeta}^n|+\rangle\otimes|+\rangle + i\eta|-\rangle\otimes|-\rangle) = \omega^k\bar{\zeta}^k(\bar{\zeta}^n|+\rangle\otimes|+\rangle + i\eta|-\rangle\otimes|-\rangle)\,,$$

where ω is the discrete torsion phase by which we have to modify the action of g_2 on the ground state in the g_1-twisted sector. Thus if we define the RR sector ground states as

$$|B_n\rangle\rangle^{(0)}_{T_{g_1}} = \omega^n\bar{\zeta}^n\left(\bar{\zeta}^n|+\rangle\otimes|+\rangle + i\eta|-\rangle\otimes|-\rangle\right)\,, \tag{6.9}$$

then $g_2|B_n\rangle\rangle^{(0)}_{T_{g_1}} = |B_{n+1}\rangle\rangle^{(0)}_{T_{g_1}}$. By summing over the ground states $\sum_n|B_n\rangle\rangle^{(0)}_{T_{g_1}}$ we see that $i\eta|-\rangle\otimes|-\rangle$ survives, if the discrete torsion phase is given by $\omega=\zeta$, and $|+\rangle\otimes|+\rangle$ survives if $\omega=\bar{\zeta}$. Thus we have constructed a brane which couples to the twisted RR sector.

We now have to construct the open strings corresponding to these branes. Since the orbifold invariant combination of branes is a superposition of the three boundary states $|B_n\rangle\rangle$, the open string spectrum has sectors which correspond to strings that start and end on the branch $|B_n\rangle\rangle$ of the orbifold invariant combination, and open strings which stretch between different branches $|B_n\rangle\rangle$ and $|B_m\rangle\rangle$. The open string which corresponds to this boundary state therefore has a 3×3 Chan-Paton matrix

$$A := \begin{pmatrix} a_{00} & a_{01} & a_{02}\\ a_{10} & a_{11} & a_{12}\\ a_{20} & a_{21} & a_{22}\end{pmatrix}\,, \tag{6.10}$$

where the diagonal elements label strings that start and end on the same brane, and the off-diagonal elements label strings which stretch between different branes. In this notation, a_{ij} labels a string which starts at brane i and ends at brane j, and a_{ji} labels a string with the opposite orientation. The orbifold group acts on the Chan-Paton matrix by conjugation

$$A \mapsto \gamma(g_k) A \gamma(g_k)^{-1} .$$

From the action of the g_i on the boundary state, and from the modular transformation of the cylinder amplitudes, we can read off the action of g_1 and g_2 on the Chan-Paton factors. They are given by

$$\gamma(g_1) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix} , \qquad \gamma(g_2) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} . \tag{6.11}$$

Thus $\gamma(g_1)$ multiplies strings stretching between different branes by an appropriate phase, and $\gamma(g_2)$ permutes strings on different branes. In principle we have to determine the action of $(-1)^F$ on the Chan-Paton factors as well, but since the branes are not orthogonal or parallel to each other this is hard to do [16].

6.3. Supersymmetry

Although the gluing conditions we have considered might seem unfamiliar, they actually describe conventional D3-branes. The most general gluing conditions for free bosons and fermions can be written as

$$\left(a_r^i + M^{ij} \tilde{a}_{-r}^j \right) \| B \rangle\!\rangle = 0 , \tag{6.12}$$

where the matrix M is some element of $\mathrm{O}(d)$, d being the number of spacetime coordinates. If M has eigenvalues different from ± 1, this can be linked to electric fields on the brane. Let us denote by M_n the gluing matrix associated to $\| B_n \rangle\!\rangle$. Then, M_n is in terms of the real coordinates given by

$$M_n = \begin{pmatrix} c_{2n} & s_{2n} & 0 & 0 & 0 & 0 \\ s_{2n} & -c_{2n} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & c_{2n} & s_{2n} \\ 0 & 0 & 0 & 0 & s_{2n} & -c_{2n} \\ 0 & 0 & c_{2n} & s_{2n} & 0 & 0 \\ 0 & 0 & s_{2n} & -c_{2n} & 0 & 0 \end{pmatrix} , \tag{6.13}$$

where we have defined

$$c_m := \cos\left(\frac{m\pi}{3}\right) \qquad \text{and} \qquad s_m := \sin\left(\frac{m\pi}{3}\right) . \tag{6.14}$$

If we diagonalise M_0, we can verify that (6.13) corresponds to a $D3$-brane. In terms of the basis

$$
\begin{aligned}
b_r^1 &= a_r^1 \, , \\
b_r^2 &= a_r^2 \, , \\
b_r^3 &= \frac{1}{\sqrt{2}} \left(a_r^3 + a_r^5 \right) \, , \\
b_r^4 &= \frac{1}{\sqrt{2}} \left(a_r^4 + a_r^6 \right) \, , \\
b_r^5 &= \frac{1}{\sqrt{2}} \left(a_r^4 - a_r^6 \right) \, , \\
b_r^6 &= \frac{1}{\sqrt{2}} \left(-a_r^3 + a_r^5 \right) \, ,
\end{aligned}
\tag{6.15}
$$

M_0 takes the form $M_0 = \mathrm{diag}(1,-1,1,-1,1,-1)$. With respect to this basis M_n is given by

$$
M_n = \begin{pmatrix}
c_{4n} & s_{4n} & 0 & 0 & 0 & 0 \\
s_{4n} & -c_{4n} & 0 & 0 & 0 & 0 \\
0 & 0 & c_{2n} & -s_{2n} & 0 & 0 \\
0 & 0 & -s_{2n} & -c_{2n} & 0 & 0 \\
0 & 0 & 0 & 0 & c_{2n} & -s_{2n} \\
0 & 0 & 0 & 0 & -s_{2n} & -c_{2n}
\end{pmatrix} . \tag{6.16}
$$

All M_n's have three eigenvalues $(+1)$ and three eigenvalues (-1). They therefore correspond to D3-branes without flux.

In terms of the b_r^i the action of g_1 is given by a rotation by an angle of $\frac{2\pi}{3}$ in the b^3-b^5 and the b^4-b^6 plane. It can be easily checked that indeed $g_1^{-k} M_n g_1^k = M_n$. The action of g_2 in the b^i basis is given by the matrix

$$
g_2 = \begin{pmatrix}
c_2 & s_2 & 0 & 0 & 0 & 0 \\
-s_2 & c_2 & 0 & 0 & 0 & 0 \\
0 & 0 & (c_1)^2 & -s_1 c_1 & s_1 c_1 & -(s_1)^2 \\
0 & 0 & s_1 c_1 & (c_1)^2 & (s_1)^2 & s_1 c_1 \\
0 & 0 & -s_1 c_1 & (s_1)^2 & (c_1)^2 & -s_1 c_1 \\
0 & 0 & -(s_1)^2 & -s_1 c_1 & s_1 c_1 & (c_1)^2
\end{pmatrix} . \tag{6.17}
$$

This matrix can be decomposed into commuting matrices g_2^H and g_2^M,

$$
g_2 = g_2^M g_2^H \, .
$$

g_2^H leaves the world-volumes of the branes described by M_n invariant

$$
(g_2^H)^{-m} M_n (g_2^H)^m = M_n \, , \tag{6.18}
$$

whereas g_2^M rotates the world-volumes

$$
(g_2^M)^{-m} M_n (g_2^M)^m = M_{n+m} \, . \tag{6.19}
$$

Explicitly, g_2^M and g_2^H are given by

$$g_2^H = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & c_1 & 0 & s_1 & 0 \\ 0 & 0 & 0 & c_1 & 0 & s_1 \\ 0 & 0 & -s_1 & 0 & c_1 & 0 \\ 0 & 0 & 0 & -s_1 & 0 & c_1 \end{pmatrix} \tag{6.20}$$

and

$$g_2^M = \begin{pmatrix} c_2 & s_2 & 0 & 0 & 0 & 0 \\ -s_2 & c_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & c_1 & -s_1 & 0 & 0 \\ 0 & 0 & s_1 & c_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & c_1 & -s_1 \\ 0 & 0 & 0 & 0 & s_1 & c_1 \end{pmatrix} . \tag{6.21}$$

With respect to the b^i basis each M_n has one Neumann and one Dirichlet direction in each of the planes b^1-b^2, b^3-b^4 and b^5-b^6, respectively. g_2^M acts as a rotation in each of these three planes.

After these preparations we are now in the position to apply the supersymmetry analysis for branes at angles of [33, 34]. As is explained there, such configurations of D3-branes preserve supersymmetry if the total angle of rotations in g_2^M is an integer multiple of 2π, as is easily checked. This shows that the above system of D3-branes is indeed supersymmetric.

7. Conclusions

We have constructed a supersymmetric stable BPS brane in an orbifold with discrete torsion which couples to twisted and untwisted RR potentials. We called the brane constructed here a permutation brane but, as explained in the previous section, in the theory of free bosons and fermions permutation boundary states do not form a new class of boundary states.

Even though we chose for the sake of concreteness to look at the g_1-twisted sector, we could have constructed this kind of brane as well for the g_2 or $g_1 g_2$-twisted sectors, and the analysis would have been identical. In each case the boundary state is a supersymmetry preserving superposition of D3-branes. We have therefore managed to construct BPS D-branes which couple to the twisted RR potentials.

Although we performed this construction explicitly for the case of a $\mathbb{Z}_3 \times \mathbb{Z}_3$ orbifold, it should be straightforward to generalise it to other $\mathbb{Z}_m \times \mathbb{Z}_m$ orbifolds with odd m; the case with even m is different and has been solved in [16]. For the sake of simplicity, we have only considered the uncompactified theory, however, compactification on tori should be an easy exercise. Another interesting test for mutual consistency would be to calculate the amplitude between a conventional brane and the brane constructed here.

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A. Fermionic zero modes

We use the following conventions for the fermionic zero modes. ψ_0^i and $\tilde{\psi}_0^i$ are defined to be raising operators, and $\psi_0^{\bar{\imath}}$ and $\tilde{\psi}_0^{\bar{\imath}}$ are lowering operators. The state annihilated by $\psi_0^{\bar{\imath}}$ we call $|-\rangle$, and the state which arises by acting with the raising operator on $|-\rangle$ we call $|+\rangle := \psi_0^i|-\rangle$. We have to build tensor products of left and right-movers and of various copies of the above algebra. Let us start with the tensor product of left and right-movers. We denote the state $|-\rangle \otimes |-\rangle$ to be the state annihilated by both $\psi_0^{\bar{\imath}}$ and $\tilde{\psi}_0^{\bar{\imath}}$. Furthermore, we define the state

$$|+\rangle \otimes |+\rangle := \psi_0^i \tilde{\psi}_0^i |-\rangle \otimes |-\rangle.$$

Due to the fact that fermions anti-commute, the order of the operators is essential. We will order the raising operators such that the tilded modes stand right of the untilded modes.

Let us now consider the case where we have to pairs of raising left moving creators and annihilators. Let us call them ψ_0^1, ψ_0^2, $\psi_0^{\bar{1}}$, $\psi_0^{\bar{2}}$. The ground state shall be denoted by $|--\rangle$. Then we define the state $|++\rangle$ to be given by

$$|++\rangle := \psi_0^1 \psi_0^2 |--\rangle = -\psi_0^2 \psi_0^1 |--\rangle\,.$$

Our conventions for the right-movers are identical. The case for more than two oscillators on the left and right moving side is analogous. With these conventions we obtain a definite sign rule which is used, wherever the RR ground states are considered.

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